

## Complexity of Automaton Identification from Given Data\*

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The question of whether there is an automaton with  $n$  states which agrees with a finite set  $D$  of data is shown to be *NP*-complete, although identification-in-the-limit of finite automata is possible in polynomial time as a function of the size of  $D$ . Necessary and sufficient conditions are given for  $D$  to be realizable by an automaton whose states are reachable from the initial state by a given set  $T$  of input strings. Although this question is also *NP*-complete, these conditions suggest heuristic approaches. Even if a solution to this problem were available, it is shown that finding a minimal set  $T$  does not necessarily give the smallest possible  $T$ .

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## 1. INTRODUCTION

*NP-Complete*

A predicate  $P(x)$  which is *NP*-complete is difficult to compute:  $P(x)$  can be computed in exponential time as a function of the size of the input  $x$ , but it is believed that polynomial time computation is not possible; see Karp (1975) and Aho *et al.* (1974).

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*Minimum Automaton Identification from Given Data is NP-Complete*

Suppose that we are given data  $D$  consisting of a finite number of observations of the  $I/O$  behavior of an unknown black box with  $I/O$  function  $b$ . The principal objective of this paper is to show that the problem of finding a minimum state finite automaton which agrees with  $D$  is “NP-complete” in the following sense: Theorem 2 (transition assignment is NP-complete) implies that the question “Is there an automaton with  $n$  states which agrees with  $D$ ?” is NP-complete.

*Nerode Algorithm*

The complexity results of this paper resulted from efforts to adapt the Nerode algorithm to the problem of automaton identification from given data. The Nerode (1958) algorithm for the problem of *automaton synthesis* yields the minimum finite state automaton which realizes a given black box function  $b$ . Automaton synthesis assumes that the entire function  $b$  is given. In particular, the Nerode algorithm assumes that means are available for determining if  $b_{\bar{u}} = b_{\bar{v}}$  for any pair  $\bar{u}, \bar{v}$  of input strings, where  $b_{\bar{u}}$  is the black box which results if  $\bar{u}$  is applied to  $b$ . The Nerode algorithm can be used with a finite amount of data if the number of states needed to realize the unknown black box is specified, and if the experiments which produce the data can be chosen.

The Ho algorithm adapts the Nerode algorithm to the synthesis of linear automata, see Zeiger (1967). Application of the Nerode approach to more general classes of automata is straightforward, e.g., see Arbib and Zeiger (1969). Arbib and Manes (1974) discuss further generalization to abstract machines in a category-theoretic framework.

*Automaton Identification from Finite Data*

The work referenced above is concerned with generalization of the Nerode algorithm to larger classes of machines. The work which led to this paper was concerned with the adaptation of the Nerode algorithm to the problem of *automaton identification from finite data*: One wishes to identify a black box which is known to be realizable by a finite (state) automaton, but the necessary number of states is not known. Only a finite amount of data is available, so it is not possible to prove  $b_{\bar{u}} = b_{\bar{v}}$ .

There are 2 variations of this problem: automaton identification from requested data, and automaton identification from given data. In the case of *requested data*, any finite number of experiments, chosen at will, can be performed on the black box which can be reset to its initial state before each experiment. The identification algorithm must choose the experiments as well as use the results of these experiments to guess a finite automaton which, hopefully, realizes the  $I/O$  function  $b$  of the unknown black box. In the case of automaton identification

from *given data*, the identification algorithm has no choice about the data, it is given.

In Gold (1972) I discuss a straightforward adaption of the Nerode algorithm, which I call *state characterization*, to the problem of automaton identification from requested data. The results of that paper were not original, but I reference it because it introduces the notation and terminology used in this paper.

Concerning automaton identification from given data, one approach is discussed by Bierman and Feldman (1972). An obvious approach is *minimum automaton identification*: Construct a finite automaton with the minimum number of states which agrees with the given data  $D$ . This approach has many desirable properties discussed in the next section, such as efficient use of data. However, it is the objective of this paper to show that the construction of a minimum state automaton which agrees with given data is, in general, computationally difficult.

## 2. SUMMARY OF COMPLEXITY RESULTS

### *Attributes of Automaton Identification Rules*

An *automaton identification rule* is a computable function  $g$  which, given data  $D$  about black box  $b$ , produces a finite automaton  $g(D)$  ( $g$  for "guess"). In the case of requested data, an automaton identification rule also generates the experiments which produce a growing sequence of data  $D_1, D_2, \dots$ . The rule will be said to have the *identification-in-the-limit* property if it can be guaranteed that for every black box  $b$  realizable by a finite automaton there is an  $i$  such that  $g(D_i), g(D_{i+1}), \dots$  are the same and realize  $b$ . In the case of given data,  $g$  will be said to have the *identification-in-the-limit* property if for every such  $b$  there is a data set  $D_b$  such that for all data sets  $D$  which include  $D_b$  the guesses  $g(D)$  are the same and realize  $b$ .

An automaton identification rule is *feasible* if  $g(D)$  always agrees with  $D$  (defined formally in next section). The stronger *minimum automaton identification* property requires that  $g(D)$  have the minimum number of states. Since  $D$  is finite, there can be nonequivalent  $g(D)$  with this property.

Suppose  $g$  has the identification-in-the-limit property. Its space, time, and data requirements are of interest. Space complexity is not discussed here. Time complexity refers to the time required to compute  $g(D)$  as a function of the size of  $D$  if the fastest algorithm is used. Presumably, polynomial time is practical and *NP*-complete is impractical.

Concerning data requirements, in a somewhat different context (Gold, 1967) I introduced the following notion:  $g$  is *optimally data efficient* if there is no  $g'$  which, for all  $b$ , correctly identifies  $b$  from as small a data set as  $g$  and sometimes smaller. I will not try to formalize this notion in the present context because I only use it to motivate the interest in minimum automaton identification:

Suppose our guessing rule tries all finite automata in order of increasing number of states, and chooses  $g(D)$  to be the first finite automaton which agrees with  $D$ . This is a minimum automaton identification rule. Also, it is an example of identification-by-enumeration. In Gold (1967) I showed that all identification-by-enumeration rules are optimally data efficient.

#### *State Characterization from Requested Data*

In the case of requested data I showed in Gold (1972) that state characterization has the following properties:

1. Minimum automaton identification.
2. Identification-in-the-limit (implied by 1).
3. Computationally trivial.

#### *State Characterization from Given Data*

The complexity results of this paper are an outgrowth of attempts to adapt state characterization to the problem of automaton identification from given data. The proposed method of adaption is straightforward: In the case of requested data, one can request the data needed by the state characterization algorithm. In the case of given data the data which is needed by the algorithm and not provided is guessed. The problem is to guess the missing data in such a way that the constructed finite automaton will be small.

Theorem 1 (Data Matrix Agreement) is the fundamental theorem of the state characterization approach to automaton identification from finite data. It gives sufficient constraints on the use of the state characterization approach to guarantee that the constructed finite automaton will agree with the data from which it was constructed.

This theorem is of interest in itself. Indeed, it is necessary to show the validity of the earlier results on the application of state characterization to the requested data problem (Gold 1972).

Furthermore, Theorem 1 assures the validity of the *timid state characterization* algorithm for given data, in the proof of Theorem 4, which has the following properties:

1. Feasible automaton identification.
2. Identification-in-the-limit (not implied by 1).
3. Polynomial time computation.

However, the main reason for including the Data Matrix Agreement Theorem in this paper is that it serves as a lemma in the proof of the principal result, Theorem 2, which says that minimum automaton identification from given data is *NP*-complete.

*Data-Time Tradeoff*

Timid state characterization is very inefficient in its use of data. So, concerning the given data problem, the results of this paper suggest that a data-time tradeoff is necessary: Identification-in-the-limit can be achieved in polynomial time at the cost of additional data being required to correctly identify the unknown black box. The most obvious approach to obtaining optimal data efficiency is computationally impractical.

However, there is still the possibility that optimal data efficiency can be achieved in polynomial time.

The *timid state characterization* approach to given data is as follows: The given data  $D$  is searched for a subset  $D_0$  such that the state characterization algorithm can be applied to  $D_0$  without having to guess missing data. If the resulting finite automaton agrees with all of  $D$  then it is taken to be the guess  $g(D)$ . Otherwise, a feasible finite automaton  $g_{\text{tab1}}(D)$  for  $D$  is constructed in the easiest way.  $g_{\text{tab1}}(D)$  is easy to compute but doesn't have the identification-in-the-limit property.

The timid state characterization algorithm uses data efficiently in the following sense: If we are lucky, and we are given just the right type of data  $D$ , then timid state characterization will obtain a correct, minimum state realization for the unknown black box from a quantity of data which is a polynomial function of the required number of states. The timid state characterization algorithm uses data inefficiently if we are not lucky and are given data which is not directly usable by the state characterization algorithm. The timid state characterization algorithm essentially ignores data which it cannot use easily.

It is straightforward to adapt state characterization to minimum automaton identification from given data if we are not interested in computation time: A backtracking algorithm can be used to guess the missing data. Varying degrees of "timidity" can be introduced to give varying data-time tradeoffs. Namely, the backtracking can be truncated at some prior time limit.

In summary, state characterization requires a certain type of data (the results of certain experiments) and so is well suited to the requested data problem: The required data is requested and a trivial computation is capable of correctly identifying the unknown black box from a small amount of data. State characterization can be adapted to the given data problem by the use of 2 types of timidity in order to reduce computation time: A tractable subset of the given data is selected and backtracking is truncated.

By means of an appropriate combination of these 2 types of timidity I believe that reasonably data efficient identification from given data should be possible with computation time asymptotically linear. Namely, let the unknown black box be fixed and suppose that we are given an enormous body of data which resulted from a not very bizarre set of experiments. It should be possible to select a small subset of the data, correctly identify the black box without much

backtracking, then make one pass through the entire body of data to check the finite automaton which was constructed from the subset.

### 3. TERMINOLOGY AND NOTATION

#### *Strings: Prefix- and Suffix-Complete*

An *alphabet*  $U$  is a finite set.  $u$  denotes an element of  $U$ ,  $\bar{u}$  denotes a finite string of elements of  $U$ , and  $\phi$  denotes the null string.  $U^*$  denotes the set of all  $\bar{u}$  including  $\phi$ , and  $U^+$  denotes the set of all  $\bar{u}$  excluding  $\phi$ . A subset  $\Sigma$  of  $U^*$  will be called *prefix-complete* if  $\bar{u} \in \Sigma$  implies all prefixes of  $\bar{u}$  are elements of  $\Sigma$  including  $\phi$ , *suffix-complete* if  $\bar{u} \in \Sigma$  implies all suffixes of  $\bar{u} \in \Sigma$  excluding  $\phi$ .

#### *Finite Automata and Black Boxes: Reachable States*

A *finite automaton* will mean a Mealy model finite state automaton with initial state specified, namely, a 6-tuple

$$FA = \langle U, S, Y, f_{tr}, f_{out}, s_0 \rangle,$$

where  $U$  is the *input alphabet*,  $S$  is the *state alphabet*,  $Y$  is the *output alphabet*,  $f_{tr}: S \times U \rightarrow S$  is the *state transition function*,  $f_{out}: S \times U \rightarrow Y$  is the *output function*, and  $s_0$  is the *initial state*. For any  $\bar{u} \in U^*$ ,  $s_{\bar{u}}$  denotes the *state reachable* by  $\bar{u}$ , namely the state which results if  $FA$  is started in state  $s_0$  and input string  $\bar{u}$  is applied. In particular,  $s_{\phi} = s_0$ . For any subset  $T$  of  $U^*$ ,  $S_T$  denotes the set of states which are reachable by some  $\bar{u} \in T$ .

A *black box* is a triple  $\langle U, Y, b \rangle$  where  $U$  is the input alphabet,  $Y$  is the output alphabet, and  $b: U^+ \rightarrow Y$  is the *I/O function* of the black box. The black box will be denoted  $b$ . The intended interpretation is that  $y = b(\bar{u})$  is the final output if  $\bar{u}$  is applied to the black box. For any  $\bar{u} \in U^*$ ,  $b_{\bar{u}}$  denotes the *black box state* which results if  $\bar{u}$  is applied to  $b$ . Namely,  $b_{\bar{u}}$  is the *I/O function* defined by

$$b_{\bar{u}}(\bar{v}) = b(\bar{u}\bar{v}) \quad \text{for all } \bar{v} \in U^+.$$

In particular,  $b_{\phi} = b$ . The *I/O function* of a finite automaton  $FA$  is a black box which will be denoted  $B(FA)$ .  $FA$  will be said to *realize*  $b$  if  $B(FA) = b$ . The black box value  $b(\phi)$  is undefined because finite automata are Mealy model.

#### *Experiments*

For any  $\bar{w} \in U^+$  the *experiment*  $e_{\bar{w}}$  is the functional

$$e_{\bar{w}}(b) = b(\bar{w}) \quad \text{for all } b,$$

Therefore,

$$e_{\bar{w}}(b_{\bar{u}}) = b(\bar{u}\bar{w}).$$

The output  $e_{\bar{w}}(b)$  is the *result* of experiment  $e_{\bar{w}}$  performed on  $b$ .  $e_{\bar{w}}(FA)$  means  $e_{\bar{w}}(B(FA))$ . The experiment  $e_{\phi}$  is undefined because finite automata are Mealy model.

### Data

*Data* (set of data, body of data) is a finite set of pairs

$$D = \{(\bar{u}_1, y_1), \dots, (\bar{u}_n, y_n)\},$$

where  $\bar{u}_i \in U^+$ ,  $y_i \in Y$ , and the  $\bar{u}_i$  are all different. Each pair  $(\bar{u}_i, y_i)$  is a *datum*.  $D(\bar{u})$  will mean  $y$  if  $(\bar{u}, y)$  is a datum of  $D$ , undefined otherwise. A black box  $B$  agrees with  $D$  if

$$b(\bar{u}_i) = y_i \quad \text{for } i = 1, \dots, n.$$

A finite automaton  $FA$  agrees with  $D$  if  $B(FA)$  agrees with  $D$ . I will use the following notation to specify  $D$ :

$$D = \begin{cases} 00 \rightarrow 2 \\ 01 \rightarrow 1. \\ 1 \rightarrow 2 \end{cases}$$

If  $D$  is *prefix-complete*, i.e., if its set of input strings is prefix-complete, then the abbreviated notation

$$D = \begin{cases} 10 \rightarrow 10 \\ 0110 \rightarrow 0111, \end{cases}$$

means

$$D = \begin{cases} 1 \rightarrow 1 \\ 10 \rightarrow 0 \\ 0 \rightarrow 0 \\ 01 \rightarrow 1 \\ 011 \rightarrow 1 \\ 0110 \rightarrow 1. \end{cases}$$

### State Characterization Matrix

A set of *test states* is a finite set of input strings

$$T = \{\bar{u}_1, \dots, \bar{u}_n\} \text{ with } \bar{u}_1 = \phi.$$

Define the set of transition states to be

$$X(T) = TU - T$$

$$= \{\bar{u}u : \bar{u} \in T, u \in U, \bar{u}u \notin T\}.$$

A set of experiments will ambiguously mean a set of non-null input strings

$$E = \{\bar{w}_1, \dots, \bar{w}_r\} \quad \text{with } \bar{w}_i \neq \phi.$$

The elements of  $E$  will sometimes be considered to be the experiments  $e_{\bar{w}_i}$  determined by the  $\bar{w}_i$ . A state characterization matrix is a triple  $\langle T, E, M \rangle$  where  $M$  is a matrix with labeled rows and columns such that

1. The rows are labeled with the elements of  $T \cup X(T)$ .
2. The columns are labeled with the elements of  $E$ .
3. Each entry of  $M$  is either an element of  $Y$  or a "hole."
4. If  $\bar{v}_i, \bar{v}_j \in T \cup X(T)$  and  $\bar{w}_i, \bar{w}_j \in E$  and  $\bar{v}_i\bar{w}_i = \bar{v}_j\bar{w}_j$  then the  $(\bar{v}_i, \bar{w}_i)$  and  $(\bar{v}_j, \bar{w}_j)$  positions in  $M$  will be called *tied*. Tied positions must have the same entry.

The data contained in  $M$  is

$$D(M) = \{(\bar{v}\bar{w}, y) : \bar{v} \in T \cup X(T), \bar{w} \in E, \text{ the } (\bar{v}, \bar{w}) \text{ entry of } M \text{ is } y \in Y\}.$$

#### 4. STATE CHARACTERIZATION FROM GIVEN DATA

##### Transition Assignment Problem $P_{\text{TRASS}}(D, T)$

Given data  $D$ , to use state characterization a set  $T$  of test states is chosen and the hypothesis is made that  $D$  agrees with some finite automaton  $FA$  whose states are reachable by  $T$ , i.e.,  $s_T = S$ . In order to construct  $FA$ , note that its input set  $U$  and output set  $Y$  are determined by  $D$ . For the present we can take its state set  $S$  to be  $T$ , although we may later find that some of these states can be identified. It is assumed that each state  $t_{\bar{u}} \in T$  will be reachable by  $\bar{u}$ , i.e.,  $s_{\bar{u}} = t_{\bar{u}}$ . So the initial state of  $FA$  must be  $s_0 = t_{\phi} \in T$ . So it only remains to construct  $f_{\text{tr}}$  and  $f_{\text{out}}$ .

The problem of constructing  $f_{\text{tr}}$  will be called the *transition assignment problem*  $P_{\text{TRASS}}(D, T)$ . Since  $t_{\bar{u}}$  is to be reachable by  $\bar{u}$ , if  $\bar{u}, \bar{u}u \in T$  then necessarily  $f_{\text{tr}}(t_{\bar{u}}, u) = t_{\bar{u}u}$ . So the transition assignment problem is to identify each transition state  $x \in X(T)$  with some test state  $t \in T$  in a way which is consistent with the data  $D$ .

If  $f_{\text{tr}}$  can be constructed it is easy to construct  $f_{\text{out}}$  so that  $FA$  has the desired property. This justifies the definition:



*Transition assignment question*  $Q_{\text{TRASS}}(D, T)$ . Given: data  $D$ , test states  $T$ . Question: Is there a finite automaton with states reachable by  $T$  which agrees with  $D$ ?

*Data Matrix*  $M(D, T, E)$ ; *Obviously Different Rows*

Given  $D$  and having chosen  $T$  we now choose a set  $E$  of experiments. The *data matrix*  $M(D, T, E)$  is the state characterization matrix with rows  $T \cup X(T)$  and columns  $E$  such that for  $\bar{u} \in T \cup X(T)$ ,  $\bar{w} \in E$  the  $(\bar{u}, \bar{w})$  entry is  $D(\bar{u}\bar{w})$  if defined by  $D$ , a hole otherwise (see Fig. 1). Given  $D, T$  it is always possible to choose  $E$  large enough so that all the data of  $D$  is contained in (the entries of)  $M(D, T, E)$ , but this is not required by the definition.

The  $\bar{u}_1$ - and  $\bar{u}_2$ -rows of  $M(D, T, E)$ , or of any state characterization matrix  $M$ , will be called *obviously different* if for some  $e \in E$  the entries  $(\bar{u}_1, e)$  and  $(\bar{u}_2, e)$  of  $M$  are different and neither is a hole. In this case  $\bar{u}_1, \bar{u}_2$  will be said to be *obviously different in*  $M$ . If  $M$  has no holes, then 2 rows are either identical or obviously different.

If we wish to construct  $FA$  in agreement with  $M$  such that the states of  $FA$  are the row labels of  $M$ , then each row of  $M$  is a vector of characteristics (attributes) of the state of  $FA$  which corresponds to its row label. The holes are "don't cares." So, 2 states of  $FA$  can't be identified if their rows of  $M$  are obviously different.

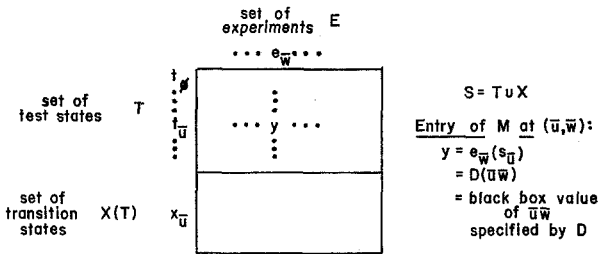


FIG. 1. Data matrix  $M(D, T, E)$ .

*Automaton*  $FA(M)$  *Constructed from State Characterization Matrix*  $M$

Let  $M$  be a state characterization matrix *with no holes* such that every  $x$ -row is identical to some  $t$ -row. The *constructed automata*  $FA(M)$  are defined non-deterministically as follows. The definition is nondeterministic, but all the constructed automata are deterministic, and all of them satisfy Theorem 1.

The input, output alphabets  $U, Y$  of  $FA(M)$  are specified by  $M$ . The state alphabet  $S$  of  $FA(M)$  is any partitioning of  $T$  into equivalence classes such that if  $\bar{u}, \bar{v} \in T$  are in the same equivalence class  $[\bar{u}] \in S$  then (1) the  $\bar{u}$ - and  $\bar{v}$ -rows of  $M$  are identical, and (2) for all  $u \in U$ , if  $\bar{u}u, \bar{v}u \in T$  then  $[\bar{u}u] = [\bar{v}u]$ , otherwise

the  $\bar{u}u$ - and  $\bar{v}u$ -rows of  $M$  are identical. The initial state  $s_0$  of  $FA(M)$  is  $[\phi]$ . For all  $\bar{u} \in T, u \in U$ , the dynamics of  $FA(M)$  are defined by

$$f_{out}([\bar{u}], u) = \begin{cases} (\bar{u}, u)\text{-entry of } M \text{ if } u \in E \\ \text{arbitrary otherwise,} \end{cases}$$

$$f_{tr}([\bar{u}], u) = \begin{cases} [\bar{v}u] \text{ if there is a } \bar{v} \in [\bar{u}] \text{ such that } \bar{v}u \in T \\ \text{otherwise any } [\bar{v}] \text{ such that } \bar{v} \in T \text{ and the } \bar{v}\text{-row of } M \\ \text{is identical to the } \bar{u}u\text{-row.} \end{cases}$$

The Data Matrix Agreement Theorem says that if  $T$  is prefix-complete and  $E$  is suffix-complete then the states of  $FA(M)$  are reachable by  $T$  and  $FA(M)$  agrees with the data in  $M$ .

*Hole Filling Problem*  $P_{HOFil}(D, T, E)$

Given data  $D$ , test states  $T$ , experiments  $E$  the *hole filling problem*  $P_{HOFil}(D, T, E)$  is to fill the holes of the data matrix  $M(D, T, E)$  to obtain a state characterization matrix  $M'$  such that every  $x$ -row is identical to some  $t$ -row. If some  $x$ -row was obviously different from every  $t$ -row in  $M(D, T, E)$  then there is no solution to the hole filling problem. Otherwise, the hole filling question "Is there a solution to  $P_{HOFil}(D, T, E)$ ?" may be difficult to decide because tied holes must be filled with the same  $y \in Y$  for  $M'$  to be a state characterization matrix. An example is shown in Fig. 2.

Suppose we are given data  $D$  and a prefix-complete set  $T$  of test states. We can choose any set of experiments such that all the data of  $D$  is in  $M(D, T, E)$  and augment it to obtain a set  $E(D, T)$  of experiments with the additional property of being suffix-complete. If we can find a solution  $M'$  to the hole filling problem  $P_{HOFil}(D, T, E(D, T))$  then  $FA(M')$  is a finite automaton with states reachable by  $T$  which agrees with  $D$ .

The converse is easy to prove. So the question  $Q_{TRASS}(D, T)$  "Is there an  $FA$  with states reachable by  $T$  which agrees with  $D$ ?" is equivalent to "Is there a

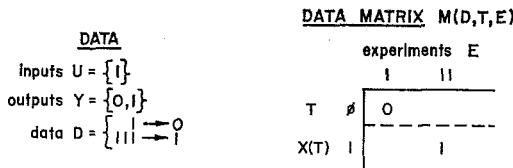


FIG. 2. Data matrix with rows which are not obviously different but can't be identical. Therefore, there is no finite automaton with 1 state which agrees with  $D$ .  $s_1$  cannot be identified with  $s_\phi$  because the 2 holes in  $M(D, T, E)$  are tied. No matter how they are filled,  $s_1$  will become obviously different from  $s_\phi$ .

solution to  $P_{\text{HoleFill}}(D, T, E(D, T))$ ?" This result will be used to prove that minimum automaton identification is *NP*-complete. Namely, Cook's prototype *NP*-complete problem will be reduced to the hole filling problem.

*Augmentation of Test States*

Still assuming that the test states  $T$  are prefix-complete, if there is no solution to the hole filling problem  $P_{\text{HoleFill}}(D, T, E(D, T))$  then there is no finite automaton with states reachable by  $T$  which agrees with  $D$ . So it is necessary to augment  $T$ .

For example, in the data matrix  $M(D, T, E(D, T))$  suppose that  $x \in X(T)$  was obviously different from all  $t \in T$  before we tried filling holes. Then we would probably add  $x$  to  $T$ . If no  $x$  was obviously different from all  $t \in T$  in the data matrix but every way of filling the holes (with tied holes being filled the same) gives  $M'$  in which some  $x$  is not identical to any of the  $t \in T$ , then it is difficult to decide how to augment  $T$ ; see Sect. 7.

In any case, if we start with a prefix-complete  $T$ , say  $T = \{\phi\}$ , and always augment  $T$  with elements of  $X(T)$ , then  $T$  will remain prefix-complete.

5. THEOREM 1. DATA MATRIX AGREEMENT

Jean-Paul Brassard constructed the example of Fig. 3 to show me that it is possible that a state characterization matrix  $M$  has no holes, and every  $x$ -row is identical to some  $t$ -row so that  $FA(M)$  can be constructed, but  $FA(M)$  doesn't agree with  $D(M)$ . In this example  $FA(M)$  is uniquely defined. It has 1 state  $s_0$  and its only output is 1. Jean-Paul Brassard suggested that in this example  $FA(M)$  does not agree with  $D(M)$  because there are gaps in  $E$ , namely,  $1 \in E$  and  $111 \in E$ , but  $11 \notin E$ . This observation led to the following theorem.

**THEOREM 1.** *Let  $\langle T, E, M \rangle$  be a state characterization matrix without holes such that every  $x$ -row is identical to some  $t$ -row,  $T$  is prefix-complete, and  $E$  is suffix-complete. Then  $FA(M)$  agrees with the data  $D(M)$  in  $M$ . Furthermore, if  $\bar{u} \in T$  then starting  $FA(M)$  in state  $s_0$  and applying  $\bar{u}$  puts  $FA(M)$  in state  $[\bar{u}]$ .*

**COROLLARY.** *Let data  $D$  and test states  $T$  be given such that  $T$  is prefix-*

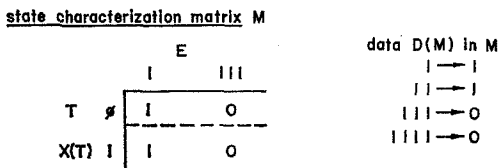


FIG. 3. Example of  $M$  such that  $FA(M)$  does not agree with  $D(M)$ .

complete. Choose any suffix-complete set  $E$  of experiments such that  $M(D, T, E)$  contains all the data in  $D$ . Then the question  $Q_{\text{TRASS}}(D, T)$  "Is there a finite automaton with states reachable by  $T$  which agrees with  $D$ ?" is equivalent to the question  $Q_{\text{HOLEFIL}}(D, T, E)$  "Can the holes of  $M(D, T, E)$  be filled in such a way that every  $x$ -row will be identical with some  $t$ -row?" Given  $D, T$  a suitable  $E$  can be found in polynomial time.

*Proof.* The proof of the corollary is straightforward and will be omitted.

The second conclusion of the theorem, that  $\bar{u} \in T$  implies state  $[\bar{u}]$  of  $FA(M)$  is reachable by  $\bar{u}$ , follows immediately from the definition of  $f_{\text{tr}}$  in  $FA(M)$ , given that  $T$  is assumed to be prefix-complete. Namely, induction can be used on the prefixes of  $\bar{u}$ .

Let  $\bar{u} \in T \cup X(T)$ ,  $\bar{w} \in E$ . It is to be proved that if  $FA(M)$  is started in  $s_0$  and  $\bar{u}\bar{w}$  is applied, then its final output will be  $y =$  the  $(\bar{u}, \bar{w})$ -entry of  $M$ . This is equivalent to saying that if  $FA(M)$  is started in  $s_{\bar{u}}$  and  $\bar{w}$  is applied, then the final output will be  $y$ .

*Case 1.*  $\bar{u} \in T$ . Then  $s_{\bar{u}} = [\bar{u}]$  by the second conclusion of the theorem. Induction will be used on the length of  $\bar{w}$ . If  $\bar{w} = u \in U$  then the output produced by applying  $u$  to  $[\bar{u}]$  of  $FA(M)$  is  $f_{\text{out}}([\bar{u}], u)$ . This is the  $(\bar{u}, u)$ -entry of  $M$  by the definition of  $f_{\text{out}}$ . So, assume that the first conclusion is proved for some  $\bar{w}$  and all  $\bar{u} \in T$ . It only remains to prove it for  $u\bar{w}$  and all  $\bar{u} \in T$ , assuming  $u\bar{w} \in E$ . (The hypothesis that  $E$  is suffix-complete implies that if  $u\bar{w} \in E$  then  $\bar{w} \in E$ , so this is a valid statement of the inductive step.)  $u, \bar{w}$  are now fixed. For all  $\bar{u} \in T$  it is to be shown that if  $FA(M)$  is started in  $s_{\bar{u}u}$  and  $\bar{w}$  is applied, then the final output is the  $(\bar{u}, u\bar{w})$ -entry of  $M$  which is the same as the  $(\bar{u}u, \bar{w})$ -entry since these 2 positions are tied. If  $\bar{u}u \in T$  this follows from the inductive hypothesis. If  $\bar{u}u \in X(T)$  let  $[\bar{v}] = s_{\bar{u}u} = f_{\text{tr}}([\bar{u}], u)$ . By definition of  $f_{\text{tr}}$ ,  $\bar{v} \in T$  and the  $\bar{v}$ -row of  $M$  is identical to the  $\bar{u}u$ -row. So it remains to prove that starting  $FA(M)$  in  $s_{\bar{u}u} = [\bar{v}]$  and applying  $\bar{w}$  gives the  $(\bar{u}u, \bar{w})$ -entry = the  $(\bar{v}, \bar{w})$ -entry of  $M$  as final output. Since  $\bar{v} \in T$  this follows from the inductive hypothesis.

*Case 2.*  $\bar{u} \in X(T)$ . It is to be shown that if  $FA(M)$  is started in  $s_{\bar{u}}$  and  $\bar{w}$  is applied the final output will be the  $(\bar{u}, \bar{w})$ -entry of  $M$ . Let  $\bar{u} = \bar{u}'u$  where  $\bar{u}' \in T$ . By the second conclusion,  $s_{\bar{u}'} = [\bar{u}']$ . So, by the definition of  $f_{\text{tr}}$ ,  $s_{\bar{u}} = f_{\text{tr}}([\bar{u}'], u) = [\bar{v}]$  where  $\bar{v} \in T$  and the  $\bar{u}$ - and  $\bar{v}$ -rows of  $M$  are identical. By the second conclusion  $s_{\bar{u}} = [\bar{v}] = s_{\bar{v}}$ . So it is to be shown that if  $FA(M)$  is started in  $s_{\bar{v}}$  and  $\bar{w}$  is applied then the final output will be the  $(\bar{u}, \bar{w})$ -entry = the  $(\bar{v}, \bar{w})$ -entry of  $M$ . Since  $\bar{v} \in T$  this has been proved in Case 1.

### 6. THEOREM 2. TRANSITION ASSIGNMENT IS NP-COMPLETE

Two input strings  $\bar{u}, \bar{v}$  will be called *obviously different w.r.t. D* if there is an experiment  $e_{\bar{w}}$  such that  $e_{\bar{w}}(s_{\bar{u}}) = D(\bar{u}\bar{w})$  and  $e_{\bar{w}}(s_{\bar{v}}) = D(\bar{v}\bar{w})$  are specified by  $D$

and  $e_{\bar{u}}(s_{\bar{u}}) \neq e_{\bar{v}}(s_{\bar{v}})$ . Given  $D, \bar{u}, \bar{v}$  it can be determined in polynomial time whether or not  $\bar{u}$  and  $\bar{v}$  are obviously different w.r.t.  $D$ .

**THEOREM 2.** *Let  $D, T$  range over pairs such that  $\text{Card}(U) = \text{Card}(Y) = 2, T = \{\phi, 1, \dots, 1^{n-1}\}$  for any  $n$ , and the test states are obviously different from each other w.r.t.  $D$ . Even with  $D, T$  restricted to these pairs,  $Q_{\text{TRASS}}(D, T)$  "Is there a finite automaton with states reachable by  $T$  which agrees with  $D$ ?" is NP-complete.*

*Minimum Automaton Identification Question  $Q_{\min}(D, n)$ .* Given: data  $D$ , positive integer  $n$ . Question: Is there a finite automaton with  $n$  states which agrees with  $D$ ?

**COROLLARY.**  $Q_{\min}(D, n)$  is NP-complete for  $\text{Card}(U) = \text{Card}(Y) = 2$ .

The proof of the corollary is omitted. Dana Angluin has found a reduction of the type used below to prove Theorem 2, but more complicated, which proves Theorem 2 to be valid even if  $D$  is restricted to prefix-complete data sets. Therefore,  $Q_{\min}(D, n)$  is NP-complete for prefix-complete  $D$ .

*Satisfiability Question Which Will Be Reduced to  $Q_{\text{TRASS}}$*

The satisfiability question is known to be NP-complete for conjunctive normal form expressions, which are of the form

$$F' = C_1 \& \dots \& C_n,$$

where the clauses  $C_i$  are of the form

$$C_i = c_{i1} \vee \dots \vee c_{in_i}$$

where the  $c_{ij}$  are literals  $z_k$  or  $\neg z_k$  where the  $z_k$  are Boolean variables. This question will be reduced to  $Q_{\text{TRASS}}(D, T)$  with  $(D, T)$  satisfying the constraints of Theorem 2. The first step is to reduce the question "Is  $F'$  satisfiable?" to the question "Is  $F$  satisfiable?" where  $F$  is a conjunctive normal form expression which satisfies the constraint that in each clause either none of the literals are complemented or all are. This reduction can be performed by replacing each clause,

$$C_i = z_{k_1} \vee \dots \vee z_{k_a} \vee \neg z_{k_{a+1}} \vee \dots \vee \neg z_{k_b},$$

by 2 clauses

$$C_{2i-1} = z_i' \vee z_{k_1} \vee \dots \vee z_{k_a},$$

$$C_{2i} = \neg z_i' \vee \neg z_{k_{a+1}} \vee \dots \vee \neg z_{k_b},$$

where  $z_1', \dots, z_n'$  are  $n$  new Boolean variables. It is easy to show that if specific values of  $z_1, \dots, z_n'$  satisfy  $F$  then the same values of  $z_1, \dots$  satisfy  $F'$ ; and if specific values of  $z_1, \dots$  satisfy  $F'$  then it's easy to choose values of  $z_1', \dots$  in order to satisfy  $F$ .

Furthermore, it will be assumed that  $F$  has the same number  $n$  of variables and clauses. This can be accomplished by adding any number  $\geq 1$  of new clauses

$$C_i'' = z_1'' \vee \dots \vee z_r'',$$

where the  $z_j''$  are any number  $\geq 1$  of new variables.

For any such  $F$  define the 2 characteristic functions

$$I_F(i, j) = \begin{cases} \text{"hole"} & \text{if } z_j \in C_i \text{ or } \neg z_j \in C_i \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_F(i) = \begin{cases} 1 & \text{if } C_i \text{ contains uncomplemented variables} \\ 0 & \text{if } C_i \text{ contains complemented variables.} \end{cases}$$

*Proof of Theorem 2*

For any  $F$  of the above form, with  $n$  variables and clauses, let  $D_F$  consist of the data in the state characterization matrix  $M_F$  which is defined in Fig. 4.

$M_F = M(D_F, T_n, E_n)$  contains the data in  $D_F$ ,  $T_n$  is prefix-complete, and  $E_n$  is suffix-complete. So, by the corollary to the Data Matrix Agreement Theorem it only remains to show that  $Q_{\text{HoleFill}}(D_T, T_n, E_n)$  "Can the holes of  $M_F$  be filled such that each  $x$ -row is the same as some  $t$ -row?" is equivalent to "Is  $F$  satisfiable?"

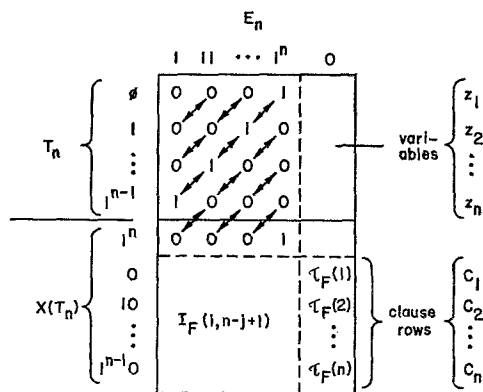


FIG. 4. State characterization matrix  $M_F$  such that the hole filling question is equivalent to "Is  $F$  satisfiable?" Double arrow denotes tied positions.

To see this, note that none of the holes of  $M_F$  are tied. The first  $x$ -row can certainly be made identical to the first  $t$ -row. In order to make the  $C_i$ -row identical to the  $z_j$ -row it is necessary to assign  $\tau_F(i)$  to the rightmost position of the  $z_j$ -row.  $\tau_F(i)$  is defined to be the value that a variable must be assigned in  $F$  in order to satisfy the  $C_i$ -clause. However, not every  $z_j$  can be assigned  $\tau_F(i)$  to satisfy  $C_i$  in  $F$ . This is simulated in  $M_F$  by the  $I_F(i, n - j + 1)$  entry in the  $j$ -th position of the  $C_i$ -row. If this entry is 0 then the  $C_i$ -row cannot be made identical to the  $z_j$ -row which has a 1 in this position. By definition of  $I_F$ , the  $C_i$ -row has a hole in this position, and can be made identical to the  $z_j$ -row, iff in  $F$  the clause  $C_i$  can be satisfied by setting  $z_j = \tau_F(i)$ .

### 7. THEOREM 3. MINIMAL SET OF TEST STATES MAY NOT BE MINIMUM

*Feasible set of test states.* A set  $T$  of test states will be called feasible for data  $D$  if there is a finite automaton with states reachable by  $T$  which agrees with  $D$ .

*State Selection Problem.* Given: data  $D$  and an oracle for  $Q_{\text{TRASS}}$ . Find: a set  $T$  of test states of minimum cardinality which is feasible for  $D$ .

One approach to the problem of constructing a minimum state automaton in agreement with  $D$  is to use a heuristic algorithm for  $P_{\text{TRASS}}$  such as state characterization with some "timidity" as discussed at the end of Sect. 2. We are still left with the state selection problem. I don't know if the state selection problem is  $NP$ -complete, but the theorem of this section shows that the obvious approach does not work:

Construct any feasible finite automaton  $FA$  for  $D$ , e.g.,  $g_{\text{tab1}}(D)$  defined in the next section, which can be constructed in polynomial time but has lots of states. Find a prefix-complete set  $T$  of input strings such that the states of  $FA$  are reachable by  $T$ . Now try removing the test states of  $T$  one at a time, in an appropriate order so that  $T$  remains prefix-complete, using the oracle to determine if  $T$  remains feasible for  $D$ . This will yield a minimal  $T$  feasible for  $D$ . But the following theorem says that a minimal feasible  $T$  is not necessarily of minimum cardinality.

**THEOREM 3.** *It is possible that set  $T$  of test states is feasible for data  $D$ , no proper subset of  $T$  is feasible for  $D$ , but there exists a  $T'$  of lower cardinality than  $T$  which is feasible for  $D$ .*

*Proof.* Consider the (prefix-complete) data

$$D = \begin{cases} 0110 \rightarrow 0111 \\ 10 \rightarrow 10. \end{cases}$$

The following 2 sets of test states are feasible for  $D$ :

$$T_1 = \{\phi, 0\}$$

$$T_2 = \{\phi, 1, 11\}.$$

It is shown below that they are minimal feasible. So  $T_2$  is obviously not minimum.

Consider experiment  $e_0$  performed on states  $s_\phi, s_1, s_{011}$  (of any finite automaton which agrees with  $D$ ).  $D$  includes the following data:

$$e_0(s_\phi) = 0; \quad e_0(s_1) = 0; \quad e_0(s_{011}) = 1.$$

So at least 2 test states are needed for a feasible  $T$ , namely  $\phi, \bar{u}$  where  $s_{\bar{u}} = s_{011}$ . This data implies  $s_1 \neq s_{011}$  so  $T = \{\phi, 1\}$  is not feasible.

Figure 6 shows that  $T_1, T_2$  are feasible. To show that  $T_2$  is minimal it is necessary to show that no pair of the 3 states of  $T_2$  can be equivalent in a feasible finite automaton for  $D$  with states reachable by  $T_2$ . If  $s_\phi = s_{11}$  or  $s_1 = s_{11}$  then  $\{\phi, 1\}$  would be a feasible set of test states for  $D$ . If  $s_\phi = s_1$ , then input 1 would take  $s_\phi$  to  $s_\phi$ , so  $s_{11} = s_\phi$ .

In order to see the difficulties involved in the state selection problem, let's follow this example a little further. Suppose we start with the hypothesis  $T = \{\phi\}$  and construct the data matrix  $M(D, T, E)$  with suffix-complete  $E$  shown in Fig. 5, which contains all the data in  $D$ . Neither of the  $x$ -rows is obviously different from the  $t$ -row. So, if we were able to determine that the hole filling problem has no solution, how would we know that we should add 0 to  $T$ , rather than 1, in order to achieve a minimum feasible  $T$  for  $D$ ?

		E									
		0		10		110		1		11	
M:	T	{	$\phi$	0	0	/	/	1	1	/	/
	X		1	0	/	/	/	/	/	/	/
	0		/	/	/	/	/	/	/	/	/

FIG. 5. Data matrix  $M$  which contains all the data in  $D$  in the proof of Theorem 3.

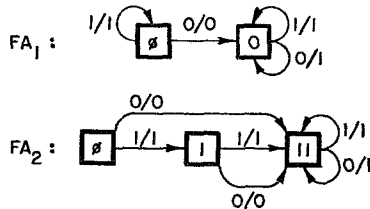


FIG. 6. Proof that  $D$  (in the proof of Theorem 3) can be realized by a finite automaton whose states are reachable by  $T_1$  or  $T_2$ .



8. THEOREM 4. AUTOMATON IDENTIFICATION-IN-THE-LIMIT WITH POLYNOMIAL TIME AND DATA

THEOREM 4. *There is an automaton identification rule  $g(D)$  with the following properties: (1) feasible ( $g(D)$  always agrees with  $D$ ), (2) identification-in-the-limit, (3)  $g(D)$  can be computed in polynomial time as a function of the size of  $D$ , and (4) for any black box  $b$  with  $p$  inputs and  $n$  state realization there is a data set  $D_b$  of size  $2n^2(p + 1)$  such that  $g(D)$  is the minimum state realization of  $b$  for all  $D \supset D_b$ . (4 implies 2.)*

*Proof.* An automaton identification rule  $g(D)$  will be demonstrated which has these 4 properties. First, an example  $g_{\text{tim}}(D)$  of timid state characterization will be defined which lacks only the first property, feasibility. That is,  $g_{\text{tim}}(D)$  will quickly and correctly identify  $b$  given  $D$  which contains the appropriate information. For other  $D$  the time to compute  $g_{\text{tim}}(D)$  is reasonable, but  $g_{\text{tim}}(D)$  may not even agree with  $D$ , much less realize  $b$ .

$g(D)$  will be obtained by changing  $g_{\text{tim}}(D)$  to  $g_{\text{tabl}}(D)$  if  $g_{\text{tim}}(D)$  doesn't agree with  $D$ .  $g_{\text{tabl}}(D)$  is easy to compute and agrees with  $D$ , but is unlikely to realize  $b$ .

*Timid State Characterization*

The following data  $D_b$  has a minimum state realization (finite automaton which agrees with  $D_b$ ) which is easy to determine and is the minimum state realization for  $b$ : Let  $T_b = \{\phi, \bar{u}_2, \dots, \bar{u}_n\}$  be a minimum prefix-complete set of test states which reach all the states  $b_{\bar{u}}$  of  $b$ . There are at most  $np$  transition states  $\bar{v}_{iu} = \bar{u}_i u \in X(T_b)$ .  $D_b$  will be constructed so that every pair  $\bar{u}_j, \bar{u}_k$  of test states are obviously different w.r.t.  $D_b$ , and each transition state  $\bar{v}_{iu}$  is obviously different from every test state  $\bar{u}_k$  except the correct one  $\bar{u}_j$ . Let experiment  $e_{\bar{w}_{jk}}$  distinguish  $b_{\bar{u}_j}$  and  $b_{\bar{u}_k}$ , i.e.,

$$e_{\bar{w}_{jk}}(b_{\bar{u}_j}) \neq e_{\bar{w}_{jk}}(b_{\bar{u}_k}).$$

Then, in order to distinguish the test states  $\bar{u}_j, \bar{u}_k$  it is sufficient for  $D_b$  to contain at most  $n(n - 1)$  datums

$$\left. \begin{aligned} \bar{u}_j \bar{w}_{jk} &\rightarrow b(\bar{u}_j \bar{w}_{jk}) \\ \bar{u}_k \bar{w}_{jk} &\rightarrow b(\bar{u}_k \bar{w}_{jk}) \end{aligned} \right\} \begin{aligned} j, k &= 1, \dots, n \\ j &< k. \end{aligned}$$

In order to distinguish each transition state  $\bar{v}_{iu}$  from the test states  $\bar{u}_k$  with  $k \neq j_i$  which are distinguishable from it, it suffices for  $D_b$  to include at most  $2n(n - 1)p$  datums

$$\left. \begin{aligned} \bar{v}_{iu} \bar{w}_{j_i k} &\rightarrow b(\bar{v}_{iu} \bar{w}_{j_i k}) \\ \bar{u}_j \bar{v}_{j_i k} &\rightarrow b(\bar{u}_j \bar{v}_{j_i k}) \end{aligned} \right\} \begin{aligned} u &\in U \\ i, k &= 1, \dots, n \\ k &\neq j_i. \end{aligned}$$

Now  $D_b$  contains information which makes it easy to construct  $f_{tr}$ . Finally, in order to specify  $f_{out}$ , at most  $np$  datums are required:

$$\bar{u}_i u \rightarrow b(\bar{u}_i u) \begin{cases} i = 1, \dots, n \\ u \in U. \end{cases}$$

The following *timid state characterization* rule for automaton identification will correctly identify  $b$  if presented  $D \supset D_b$ : Given any  $D$ , in order to construct  $g_{tim}(D)$  initialize  $T = \{\phi\}$ . If some  $x \in X(T)$  is obviously different from all  $t \in T$ , then add  $x$  to  $T$ . Continue until  $T_D$  is obtained such that for each  $x \in X(T_D)$  there is a  $t \in T_D$  such that  $t$  is not obviously different from  $x$ .  $g_{tim}(D)$  is now defined by

$$S = T_D,$$

$$\text{for } t \in T_D, u \in U : f_{tr}(t, u) = \begin{cases} tu \text{ if } tu \in T_D \\ \text{otherwise any } t' \in T_D \text{ such that } t' \text{ is not} \\ \text{obviously different from } tu \text{ w.r.t. } D, \end{cases}$$

$$\text{for } t \in T_D, u \in U : f_{out}(t, u) = \begin{cases} D(tu) \text{ if specified by } D \\ \text{arbitrary otherwise.} \end{cases}$$

Actually, in the construction of  $D_b$  it is necessary to use some care in choosing  $T_b$ . Otherwise it is possible that  $D \supset D_b$  but the set  $T_D$  of obviously different test states found by timid state characterization is not  $T_b$ .  $T_D$  will certainly be feasible for  $b$ , but  $D_b$  may not contain the appropriate information to uniquely determine the correct  $f_{tr}$ .

Let us constrain the construction of  $T_b$  as follows. First specify the rule  $g_{tim}(D)$  precisely. In particular, in the construction of  $T_D$  it is necessary to state in what order the transition states  $X(T)$  will be checked to see if they should be added to  $T$ . Now choose  $T_b$  by this same rule: Start with  $T = \{\phi\}$ . Check each  $x \in X(T)$  in the same order as used in the definition of  $g_{tim}$  to see if  $b_x$  is different from all  $b_t$  with  $t \in T$ . If so, add  $x$  to  $T$ . The resulting  $T_b$  can be used as above to define  $D_b$ , and  $g_{tim}$  will have the stated properties.

*Feasible Automaton Identification by Table Construction*

For any data  $D$  let  $b_{tabl}^D$  be the black box

$$b_{tabl}^D(\bar{u}) = \begin{cases} D(\bar{u}) & \text{if specified by } D \\ y_0 & \text{otherwise,} \end{cases}$$

where  $y_0 \in Y$  is any output element. Define  $g_{tabl}(D)$  to be the minimum state realization of  $b_{tabl}^D$ .

*Feasible Modification of Timid State Characterization*

$$g(D) = \begin{cases} g_{\text{tim}}(D) & \text{if } g_{\text{tim}}(D) \text{ agrees with } D \\ g_{\text{tabl}}(D) & \text{otherwise.} \end{cases}$$

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